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Algebraic properties of Rogers–Szegö functions: I. Applications in quantum optics

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Abstract
By means of a well-established algebraic framework, Rogers–Szegö functions associated with a circular geometry in the complex plane are introduced in the context of q-special functions, and their properties are discussed in detail. The eigenfunctions related to the coherent and phase states emerge from this formalism as infinite expansions of Rogers–Szegö functions, the coefficients being determined through proper eigenvalue equations in each situation. Furthermore, a complementary study on the Robertson–Schrödinger and symmetrical uncertainty relations for the cosine, sine and nondeformed number operators is also conducted, corroborating, in this way, certain features of q-deformed coherent states.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Ramanujan's work and subsequent studies on the q-special functions certainly represent an important chapter within the several astonishing achievements reached during recent decades in mathematics [1–10]. Another particular but no less remarkable research branch developed in parallel by mathematicians and physicists is focused on quantum groups and/or q-deformed algebras [11, 12]. Since q-deformed algebras encompass the description of a wide variety of symmetries, if one compares with those studied in the standard Lie algebras, it becomes natural to employ this powerful mathematical tool to investigate certain complex symmetries associated with nontrivial physical systems in an appropriate way. Hence, several contributions have appeared in the literature, from time to time, presenting a plethora of original results directly related to specific problems originating from solvable statistical mechanics models [13], quantum inverse scattering theory [14], nuclear physics [15, 16], molecular physics [17], some q-deformed extensions of quantum mechanics [18–22] and quantum optics with emphasis on coherent states [23–30], as well as recent applications in trapped ions by laser fields [31] and also in the Jaynes–Cummings model [32]. Furthermore, it is worth mentioning that this range of possible applications can also be extended to deformed superalgebras [33], knot theories [34] and non-commutative geometries [35].

In this scenario, there are approaches that connect q-special functions and q-deformed algebras which deserve to be elucidated because they exhibit a unifying quantum-algebraic framework for the realizations and/or representations of those algebras. For example, within the context of Lie algebras and their q-analogues, Feinsilver [36] has discussed how the three-term recurrence relations related to orthogonal polynomials can be used to obtain certain realizations in terms of raising and lowering operators. Floreanini and Vinet [37] showed that suitable operators acting on vector spaces of functions in one complex variable can be considered as possible realizations of the $\mathfrak{sl}_q(2)$ and q-oscillator algebras. Pursuing this formal line of theoretical investigation, some authors [38–40] also presented significant contributions for a special class of orthogonal polynomials, which are recognized in the literature as Rogers–Szegö (RS) and Stieltjes–Wigert (SW) polynomials [41–43]. The mathematical motivation for this specific choice of polynomials is directly associated with the q-oscillator algebra, namely, both the polynomials are viewed as concrete representations of the Iwata–Arik–Coon–Kuryshkin (IACK) algebra [20], and their respective realizations are expressed by means of Jackson's q-derivative [44, 45]. In addition, while the SW polynomials are orthogonalized on the full real line, the RS polynomials are defined upon the complex plane and orthogonalized on the unit circle through a particular measure [40, 43]. This fundamental feature intrinsic to the RS polynomials has a potential connection with angular representations in quantum mechanics, such a fact being interpreted by us as an effective gain in the debate on the polar decomposition of the annihilation operator in quantum optics [46–50].

The main goal of this paper is to present a consistent algebraic framework based on a particular set of q-special functions which allows us to go further in our comprehension of the phase operator problem and its different representations in quantum mechanics. For this purpose, we first review certain essential mathematical properties associated with the RS polynomials which permit us to establish two new (as far as we know) integral representations for such polynomials involving the finite q-Pochhammer symbol and the SW polynomials. In the following, we define our object of study (here named RS functions) through a product of two complex functions with distinct essential features, namely $\Psi_n^{\mathrm{RS}}(z;q) := \mathscr{R}_n(z;q)\mathscr{M}(z;q)$. Indeed, while the first one is a RS polynomial (at least of the normalization coefficient), the second one is responsible for the corresponding weight function which is connected with the decomposition of the Szegö measure in the complex plane. An immediate consequence of this particular sort of definition refers to its inherent orthogonality property, that is, it preserves the orthogonality relation verified for the RS polynomials. Another important property concerns the completeness relation for the RS functions that is presented in this context via a bilinear kernel [38]. The q-calculus framework is then employed to carry out a careful analysis of each aforementioned complex function, and the results originated from this analysis are used to derive the q-differential forms of the lowering, raising and number operators. It is worth stressing that the algebraic approach developed here for the RS functions leads us to obtain, in principle, not only an alternative representation for the IACK algebra but also an inherent realization.

The second part of this paper is focused basically on the construction process of coherent and phase states in accordance with the quantum-algebraic framework previously discussed. So, our first application has as reference guide the mathematical approach developed in [51] for an important class of coherent states, namely, those obtained from a determined eigenvalue equation for a given annihilation operator [23–31]. The eigenfunctions derived from this particular procedure are then expressed as an infinite expansion in terms of the RS functions whose coefficients satisfy a set of mathematical prerequesites that leads us to obtain, as a by-product, the excitation probability distribution for the q-deformed coherent state. Expressing in a clearer way, once the physical system of interest can be initially prepared in the q-deformed coherent state, this result allows us to obtain the excitation probability distribution (here labelled by the degree of excitation n of the q-deformed harmonic oscillator) for such a system. Furthermore, we discuss in detail some intrinsic properties of this definition. For the phase states and their connections with the angular representations in quantum mechanics, we have constructed the q-deformed eigenstates for the cosine and sine operators following the Carruthers–Nieto approach [47], and also presented the orthogonality and completeness relations for each situation. To complete this work, we have applied our results in order to calculate, via q-deformed coherent states, certain mean values associated with the cosine and sine operators which allow us to carry out a detailed study on the Robertson–Schrödinger and symmetrical uncertainty relations.

This paper is structured as follows. In section 2, we establish a few essential mathematical properties and also derive two additional integral representations for the RS polynomials. In section 3, we introduce the RS functions $\left\{\Psi_n^{\mathrm{RS}}(z;q)\right\}_{n\in\mathbb{N}}$ through the product of two basic complex functions whose distinct characteristics lead us to obtain a wide set of results which constitutes our quantum-algebraic framework. Section 4 is dedicated to the construction process of q-deformed coherent states related to the RS functions, where certain properties (for instance, the overlap probability and completeness relation) are discussed in detail. Moreover, we obtain in section 5 the respective eigenstates of the cosine and sine operators, as well as presenting some relevant results for each situation. The discussion on the Robertson–Schrödinger and symmetrical uncertainty relations involving such operators is presented in section 6. Finally, section 7 contains our summary and conclusions.

2. Explanatory notes on the RS polynomials

In order to make the presentation of this section more self-contained, let us initially review certain essential mathematical prerequisites of the RS polynomials, for then establishing, subsequently, two integral representations which permit us to connect such polynomials with the finite q-Pochhammer symbol and the SW polynomials. It is important to emphasize that the basic notation employed in our exposition follows some well-known textbooks on special functions, where, in particular, the q-series or Eulerian series are introduced within the context of the theory of partitions and/or basic hypergeometric series [7–10].

Definition. *Let* $\{\mathscr{H}_n(z;q)\}_{n\in\mathbb{N}}$ *designate a set of polynomials with* $0 < q < 1$ *and* $z \in \mathbb{C}$. *In particular, the RS polynomials are defined through the finite series [41, 42]*

$$\mathscr{H}_n(z;q) := \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q z^k, \tag{1}$$

where the q-binomial coefficients (also known as Gaussian polynomials)

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}} = \frac{[n]_q!}{[k]_q![n-k]_q!}$$

are expressed in terms of the finite q-Pochhammer symbol

$$(a;q)_n := \prod_{j=0}^{n-1}(1-aq^j) \qquad (a \in \mathbb{C})$$

or written as a function of the q-factorial $[n]_q! := (1-q)^{-n}(q;q)_n$.

It is worth noting that the generating function associated with $\mathscr{H}_n(z;q)$ is given by the specific Eulerian series $G(w,z;q) := [(w;q)_\infty(wz;q)_\infty]^{-1}$ for $|wz| < 1$. Thus, for future use in the text, in order to establish a consistent proof of this statement, let us first consider the particular case $G(w,0;q) = [(w;q)_\infty]^{-1}$. In this situation, the Cauchy theorem states that, for $|w| < 1$ and $0 < q < 1$, $G(w,0;q)$ is reduced to the infinite series [7]

$$G(w,0;q) = \sum_{n\in\mathbb{N}} \frac{w^n}{(q;q)_n} \qquad (w \in \mathbb{C}).$$

This result leads us to propose that $G(w,z;q)$ admits a similar expression, where now the summand is multiplied by the coefficient $\mathscr{A}_n(z;q)$,

$$G(w,z;q) = \sum_{n\in\mathbb{N}} \mathscr{A}_n(z;q)\frac{w^n}{(q;q)_n}. \tag{2}$$

Indeed, equation (2) has the essential features that we need for our purposes since the infinite product $G(w,z;q)$ is uniformly convergent for all q inside $|wz| \leqslant 1-\varepsilon$, and therefore it defines a function of w and z analytic in $|wz| < 1$. Furthermore, the identity [43]

$$G(qw,z;q) = (1-w)(1-qw)G(w,z;q)$$

allows us to show that $\{\mathscr{A}_n(z;q)\}_{n\in\mathbb{N}}$ satisfies a three-term recurrence relation

$$\mathscr{A}_{n+1}(z;q) = (1+z)\mathscr{A}_n(z;q) - (1-q^n)z\mathscr{A}_{n-1}(z;q), \tag{3}$$

where $\mathscr{A}_0(z;q) = 1$ and $\mathscr{A}_1(z;q) = 1+z$. The remaining terms for $n \geqslant 2$ determine a set of polynomials expressed explicitly in terms of z and q, whose closed formula coincides exactly with equation (1); consequently, $\mathscr{A}_n(z;q) \equiv \mathscr{H}_n(z;q)$.

Next, let us introduce some properties related to the RS polynomials where special attention will be paid to their orthogonality property. Adopting a particular parametrization for the complex variable z, Szegö [42] showed that $\{\mathscr{H}_n(z;q)\}_{n\in\mathbb{N}}$ can be orthogonalized on the circle through a specific measure which coincides with the Jacobi ϑ_3-function evaluated at continuous arguments [52], that is,

$$\int_{-\pi}^{\pi} \mathscr{H}_m\big(-q^{-\frac{1}{2}}\mathrm{e}^{-\mathrm{i}\varphi};q\big)\mathscr{H}_n\big(-q^{-\frac{1}{2}}\mathrm{e}^{\mathrm{i}\varphi};q\big)\vartheta_3\left(\frac{\varphi}{2}\middle|q^{\frac{1}{2}}\right)\frac{\mathrm{d}\varphi}{2\pi} = \frac{(q;q)_n}{q^n}\delta_{m,n}. \tag{4}$$

Subsequently, Carlitz [43] generalized such an equation by fixing the integration measure and considering different arguments of $\mathscr{H}_n(z;q)$. Since then, different authors have worked on this theme and showed some interesting peculiarities of the RS polynomials. For example, Macfarlane [18] has discussed the quantum group $SU_q(2)$ through a mathematical procedure that resembles the approach developed by Schwinger [53] for the quantum theory of angular momentum. In particular, the author showed how the coordinate representation of the q-deformed harmonic oscillator can be used in order to obtain a wavefunction which is expressed in terms of RS polynomials. Moreover, Atakishiyev and Nagiyev [38] derived an important orthogonality relation on the full real line for such polynomials, and also established a special link with the SW polynomials by means of a Fourier transform. Recently, Galetti and coworkers [40] have shown didactically both the orthogonality relations for the RS and SW polynomials, as well as obtained the explicit realizations of the raising and lowering operators for each case[1]; in addition, the authors also proposed a Wigner function related to the RS polynomials which leads us to determine a set of well-behaved marginal distribution functions with compact support for the angle and action variables.

The next property to be discussed allows us to establish a connection between the finite q-Pochhammer symbol and the RS polynomials. For this task let us initially express, by means of the Cauchy theorem, $(a;q)_n$ as a sum involving finite powers of the complex variable a, namely

$$(a;q)_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{1}{2}k(k-1)}(-a)^k.$$

This result is extremely important in our considerations since it leads us to verify that

$$\int_{-\pi}^{\pi} \mathscr{H}_n\big(-q^{-\frac{1}{2}}a^r \mathrm{e}^{\mathrm{i}s\varphi};q\big)\vartheta_3\big(u\varphi\big|q^{\frac{2u^2}{s^2}}\big)\frac{\mathrm{d}\varphi}{2\pi} = (a^r;q)_n, \tag{5}$$

where r is an arbitrary power of a and $\frac{s}{2u}$ is an integer number—note that the right-hand side of equation (5) does not depend on s and u in this situation. Thus, if one substitutes $r=1$ in such a case, this identity can be considered as a possible integral representation for the finite q-Pochhammer symbol. Another interesting additional property consists in establishing an integral representation for the RS polynomials through an adequate transformation kernel $\mathcal{P}_r(\omega;q)$, that is,

$$\int_0^{\infty} (-q^r z\omega;q)_n \mathcal{P}_r(\omega;q)\mathrm{d}\omega = \mathscr{H}_n(z;q) \tag{6}$$

with

$$\mathcal{P}_r(\omega;q) := \frac{m}{\sqrt{\pi}}\omega^{r-\frac{3}{2}} \exp\left[-\frac{1}{4m^2}\left(r-\frac{1}{2}\right)^2 - m^2\ln^2(\omega)\right] \qquad (m\in\mathbb{R})$$

and $q=\exp\left(-\frac{1}{2m^2}\right)$. In order to demonstrate such an equation, it is sufficient to know that

$$\int_0^{\infty} \omega^k \mathcal{P}_r(\omega;q)\,\mathrm{d}\omega = q^{-\frac{1}{2}k(k-1)-rk} \tag{7}$$

for the following proper parametrization: $\omega=\exp\left(\frac{x}{m^2}\right)$ $(-\infty<x<\infty)$. It is worth noting that $\mathcal{P}_r(\omega;q)$ not only encompasses certain particular cases studied by Carlitz [43], but also can be used to orthogonalize the SW polynomials.

Let us derive now a last integral representation for the RS polynomials which has, as an integrand, the product of the SW polynomials

$$\mathscr{G}_n(q^{n+r}z\omega;q) := \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)}(q^{n+r}z\omega)^k \tag{8}$$

and the transformation kernel

$$\mathcal{Y}_r(\omega;q) := \frac{m}{\sqrt{2\pi}}\omega^{\frac{r}{2}-1}\exp\left[-\frac{r^2}{8m^2} - \frac{m^2}{2}\ln^2(\omega)\right].$$

In this situation, if one considers the parametrizations used in the previous case for q and ω, it is easy to show that

$$\int_0^{\infty} \omega^k \mathcal{Y}_r(\omega;q)\,\mathrm{d}\omega = q^{-k(k+r)}, \tag{9}$$

[1] It is worth mentioning that Ismail and Rahman [39] slightly modified the argument of $\mathscr{H}_n(z;q)$ (namely, $z\to q^{-1/2}z$) with the aim of deriving the respective q-differential forms related to the raising and lowering operators. This particular procedure has then produced certain ladder operators for the RS polynomials whose formal expressions differ from those obtained in [40].

which leads us to validate the integral representation

$$\int_0^\infty \mathscr{G}_n(q^{n+r} z\omega; q)\mathcal{Y}_r(\omega; q)\, \mathrm{d}\omega = \mathscr{H}_n(z; q). \tag{10}$$

In such a case, contrasting with the relation $\mathscr{G}_n(z; q) = \mathscr{H}_n(z; q^{-1})$, the transformation kernel $\mathcal{Y}_r(\omega; q)$ permits us to identify a new link between the SW and RS polynomials through the integral representation (10). Note that equations (6) and (10) represent, in particular, two new additional results within a wide variety of formal properties obtained by Carlitz [43] for $\{\mathscr{H}_n(z; q)\}_{n\in\mathbb{N}}$. Next, we will investigate a special family of q-orthogonalized functions (here named as the RS functions) with the aim of obtaining a set of mathematical properties which leads us to formally characterize its associated algebraic structure.

3. Algebraic properties of the RS functions

In this section, we establish some preliminary mathematical results inherent to the RS functions. As a first step, we obtain a decomposition formula for the Szegö measure in the complex plane which leads us to define a weight function $\mathscr{E}(\varphi; q)$ related to our object of study. In the following step, we construct explicitly the realizations of raising and lowering operators for such functions by means of a specific definition of Jackson's q-derivative, as well as discuss the implications of this algebraic approach.

3.1. Preliminaries

The measure employed in equation (4) to orthogonalize the RS polynomials admits an expression of the form

$$\vartheta_3\left(\frac{\varphi}{2}\middle| q^{\frac{1}{2}}\right) = \sum_{\ell\in\mathbb{Z}} q^{\frac{1}{2}\ell^2}\mathrm{e}^{\mathrm{i}\ell\varphi} = 1 + 2\sum_{\ell\in\mathbb{N}^*} q^{\frac{1}{2}\ell^2}\cos(\ell\varphi). \tag{11}$$

Since equation (11) defines a strictly positive function on the interval $\varphi \in [-\pi, \pi]$ for any $q \in (0, 1)$, let us properly employ the *addition formula* [52]

$$\vartheta_3(x + y|\mathfrak{q})\vartheta_3(x - y|\mathfrak{q})\vartheta_3^2(0|\mathfrak{q}) = \vartheta_3^2(x|\mathfrak{q})\vartheta_3^2(y|\mathfrak{q}) + \vartheta_1^2(x|\mathfrak{q})\vartheta_1^2(y|\mathfrak{q})$$

with 2π-period for $x = y = \frac{\varphi}{4}$ and $\mathfrak{q} = q^{\frac{1}{2}}$, in order to obtain

$$\vartheta_3\left(\frac{\varphi}{2}\middle| q^{\frac{1}{2}}\right)\vartheta_3^3\left(0\middle| q^{\frac{1}{2}}\right) = \vartheta_3^4\left(\frac{\varphi}{4}\middle| q^{\frac{1}{2}}\right) + \vartheta_1^4\left(\frac{\varphi}{4}\middle| q^{\frac{1}{2}}\right).$$

So, if one considers the complex function (up to a phase factor)

$$\mathscr{E}(\varphi; q) := \left[\vartheta_3\left(0\middle| q^{\frac{1}{2}}\right)\right]^{-\frac{3}{2}}\left[\vartheta_3^2\left(\frac{\varphi}{4}\middle| q^{\frac{1}{2}}\right) + \mathrm{i}\vartheta_1^2\left(\frac{\varphi}{4}\middle| q^{\frac{1}{2}}\right)\right], \tag{12}$$

it is immediate to verify that (11) can also be written as a product of $\mathscr{E}(\varphi; q)$ by its respective complex conjugate. Figure 1 shows the plots of the equipotential curves related to (*a*) $\mathrm{Re}\,[\mathscr{E}(\varphi; q)]$ and (*b*) $\mathrm{Im}\,[\mathscr{E}(\varphi; q)]$ for the variables φ and q within the intervals $[-\pi, \pi]$ and $(0, 1)$, respectively. Note that, in particular, plot (*a*) presents a peak at the point $\varphi = 0$ and $q \to 1$, this behaviour being the same of that observed in our numerical investigations for the equipotential curves associated with the Szegö measure. In counterpart, plot (*b*) has two symmetric peaks located at the points $\varphi = -\pi, \pi$ and $q \to 0$, such distinct behaviour being now explained due to the presence of the Jacobi ϑ_1-function in equation (12). Consequently, the product $\mathscr{E}(\varphi; q)\mathscr{E}^*(\varphi; q)$ not only retains the pattern verified in plot (*a*) but also confirms

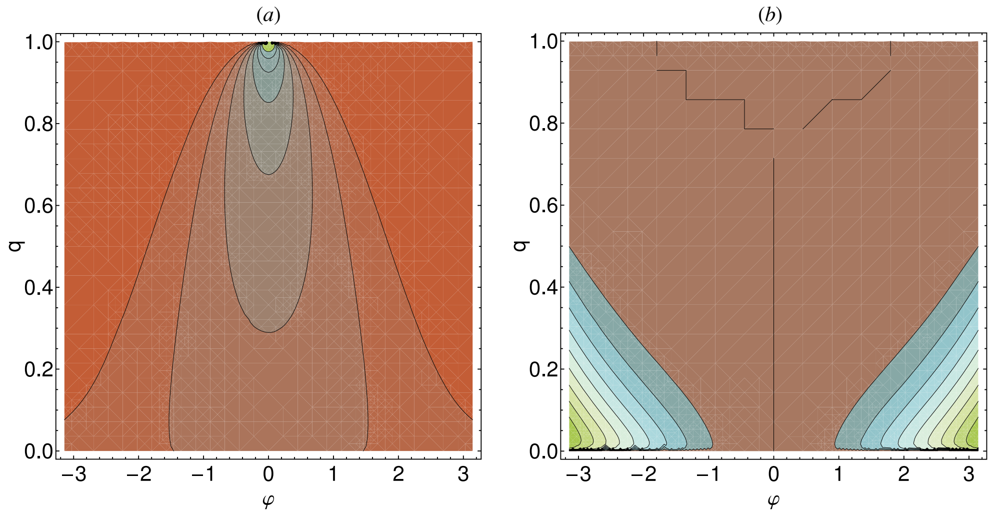

Figure 1. Plots of equipotential curves associated with the (a) real and (b) imaginary parts of the complex function $\mathscr{E}(\varphi; q)$ for $\varphi \in [-\pi, \pi]$ and $q \in (0, 1)$. The different patterns observed in both pictures are directly related to the distinct behaviours of the Jacobi ϑ_3- and ϑ_1-functions used to decompose the Szegö measure in the product $\mathscr{E}(\varphi; q)\mathscr{E}^*(\varphi; q)$.

an important property for this integration measure, that is, its decomposition into the complex plane. Next, we define properly our object of study.

Definition. *Let us initially introduce the RS functions by means of the product*

$$\Psi_n^{\mathrm{RS}}(z; q) := \left[\frac{q^n}{2\pi(q; q)_n}\right]^{\frac{1}{2}} \mathscr{H}_n(z; q)\mathscr{M}(z; q) = \mathscr{R}_n(z; q)\mathscr{M}(z; q), \tag{13}$$

where $\mathscr{R}_n(z; q)$ denotes the RS polynomials (at least of the normalization coefficient), and

$$\mathscr{M}(z; q) = \left[\mathfrak{F}\left(-q^{-\frac{1}{2}}; q\right)\right]^{-\frac{3}{2}}[\mathfrak{F}^2(z; q) + \mathrm{i}\mathfrak{G}^2(z; q)]$$

represents a weight function with

$$\mathfrak{F}(z; q) = \sum_{\ell \in \mathbb{Z}} (-\mathrm{i})^{\ell} q^{\frac{1}{2}\ell(\ell+\frac{1}{2})} z^{\frac{\ell}{2}}, \tag{14}$$

$$\mathfrak{G}(z; q) = -\sum_{\ell \in \mathbb{Z}} \mathrm{i}^{\ell+\frac{1}{2}} q^{\frac{1}{2}(\ell+\frac{1}{2})(\ell+1)} z^{\frac{1}{2}(\ell+\frac{1}{2})}. \tag{15}$$

Thus, for $z = -q^{-\frac{1}{2}}\mathrm{e}^{\mathrm{i}\varphi}$ fixed, it is immediate to see that $\mathscr{M}\left(-q^{-\frac{1}{2}}\mathrm{e}^{\mathrm{i}\varphi}; q\right) \equiv \mathscr{E}(\varphi; q)$ since equations (14) and (15) coincide, respectively, with the ϑ_3- and ϑ_1-functions. In addition, this particular parametrization also permits us to verify that

$$\int_{-\pi}^{\pi} \Psi_m^{RS*}\left(-q^{-\frac{1}{2}}\mathrm{e}^{\mathrm{i}\varphi}; q\right)\Psi_n^{\mathrm{RS}}\left(-q^{-\frac{1}{2}}\mathrm{e}^{\mathrm{i}\varphi}; q\right) \mathrm{d}\varphi = \delta_{mn}. \tag{16}$$

Hence, $\left\{\Psi_n^{RS}(z; q)\right\}_{n\in\mathbb{N}}$ features a set of well-defined functions in the complex plane which are orthogonalized on the unit circle.

The proof of the completeness relation associated with $\{\Psi_n^{\mathrm{RS}}(z;q)\}_{n\in\mathbb{N}}$ follows basically the formal mathematical treatment sketched in [5, 38]. For this task, we first define the bilinear kernel

$$K_\varepsilon(w,z;q) := \sum_{n\in\mathbb{N}} \varepsilon^n \Psi_n^{\mathrm{RS}*}(w;q)\Psi_n^{\mathrm{RS}}(z;q) \qquad |\varepsilon| < 1 \tag{17}$$

which can be evaluated by means of the auxiliary relation [7]

$$\sum_{n\in\mathbb{N}} \mathscr{H}_n(w^*;q)\mathscr{H}_n(z;q)\frac{(q\varepsilon)^n}{(q;q)_n} = \frac{(q^2\varepsilon^2 w^* z;q)_\infty}{(q\varepsilon w^* z, q\varepsilon w^*, q\varepsilon z, q\varepsilon;q)_\infty},$$

namely[2]

$$K_\varepsilon(w,z;q) = \frac{(q^2\varepsilon^2 w^* z;q)_\infty \mathscr{M}^*(w;q)\mathscr{M}(z;q)}{2\pi\,(q\varepsilon w^* z, q\varepsilon w^*, q\varepsilon z, q\varepsilon;q)_\infty}.$$

Note that for $z = -q^{-\frac{1}{2}}\mathrm{e}^{\mathrm{i}\varphi}$ and $w = -q^{-\frac{1}{2}}\mathrm{e}^{\mathrm{i}\beta}$, $K_\varepsilon(w,z;q)$ is expressed as

$$K_\varepsilon(\beta,\varphi;q) = \frac{\left(q\varepsilon^2\mathrm{e}^{\mathrm{i}(\varphi-\beta)};q\right)_\infty \mathscr{E}^*(\beta;q)\mathscr{E}(\varphi;q)}{2\pi\left(\varepsilon\,\mathrm{e}^{\mathrm{i}(\varphi-\beta)}, -q^{\frac{1}{2}}\varepsilon\,\mathrm{e}^{-\mathrm{i}\beta}, -q^{\frac{1}{2}}\varepsilon\,\mathrm{e}^{\mathrm{i}\varphi}, q\varepsilon;q\right)_\infty}. \tag{18}$$

Besides, if one takes into account the orthogonality relation (16), it is immediate to show that equation (18) obeys the following properties:

$$\int_{-\pi}^{\pi} K_\varepsilon(\beta,\varphi;q)\Psi_n^{\mathrm{RS}}\left(-q^{-\frac{1}{2}}\mathrm{e}^{\mathrm{i}\beta};q\right)\,\mathrm{d}\beta = \varepsilon^n\Psi_n^{\mathrm{RS}}\left(-q^{-\frac{1}{2}}\mathrm{e}^{\mathrm{i}\varphi};q\right), \tag{19}$$

$$\int_{-\pi}^{\pi} K_\varepsilon(\beta,\varphi;q)K_{\varepsilon'}(\gamma,\beta;q)\,\mathrm{d}\beta = K_{\varepsilon\varepsilon'}(\gamma,\varphi;q). \tag{20}$$

Thus, any well-behaved (or at least piecewise continuous) function $F(z)$ can now be properly expanded in terms of the complete set of functions $\{\Psi_n^{\mathrm{RS}}(z;q)\}_{n\in\mathbb{N}}$.

Next, we discuss certain relevant additional points associated with the definition proposed for $\Psi_n^{\mathrm{RS}}(z;q)$, which will be useful in the descriptive process of its formal properties.

(i) Guided by the analogy with the usual harmonic oscillator (HO) on the line, where the normalized wavefunction [54, p 151]

$$\Psi_n^{\mathrm{HO}}(x;\alpha) = \left(\frac{\alpha}{\pi^{1/2}2^n n!}\right)^{\frac{1}{2}} H_n(\alpha x)e^{-\frac{1}{2}(\alpha x)^2} \qquad (n\in\mathbb{N})$$

is written in terms of the Hermite polynomials $H_n(\alpha x)$ and the Gaussian weight function $\exp\left[-\frac{1}{2}(\alpha x)^2\right]$ (note that α is the HO width and also acts as a controlling parameter in this case), we may guess that (13) plays a similar role and the angular density function $\left|\Psi_n^{\mathrm{RS}}\left(-q^{-\frac{1}{2}}\mathrm{e}^{\mathrm{i}\varphi};q\right)\right|^2$ is, as a matter of fact, a good candidate in describing a phase distribution for a q-deformed HO, with q being a parameter that controls the distribution width, and is therefore responsible for *squeezing effects* [40]. In order to reinforce such

[2] The notation $(a_1,a_2,\ldots,a_r;q)_\infty \equiv (a_1;q)_\infty\cdot(a_2;q)_\infty\cdot\ldots\cdot(a_r;q)_\infty$ here employed represents the generalized q-Pochhammer symbol with $\{a_1,a_2,\ldots,a_r\}\in\mathbb{C}$.

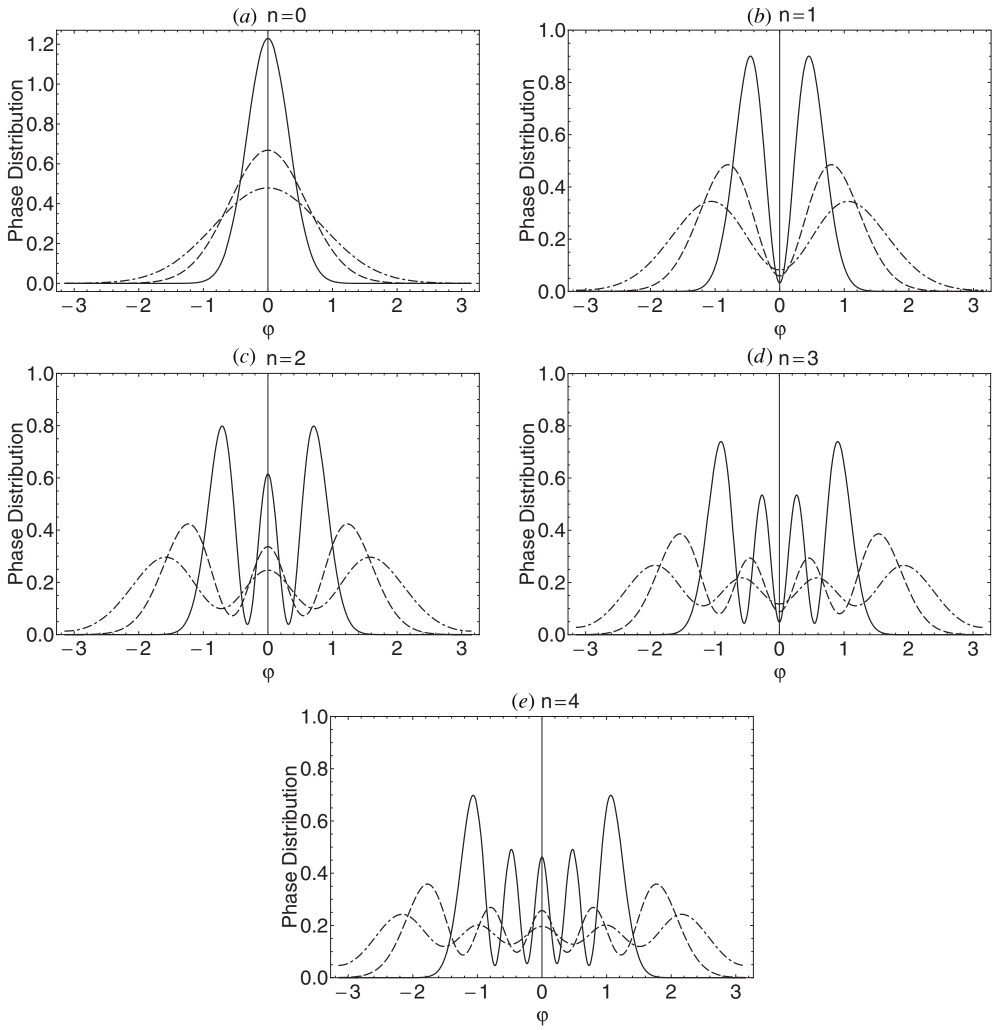

Figure 2. Plots of $\left|\Psi_n^{\mathrm{RS}}(-q^{-\frac{1}{2}}\mathrm{e}^{\mathrm{i}\varphi};q)\right|^2$ as a function of the angle $\varphi \in [-\pi,\pi]$ with $n \in [0,4]$ and different values of q, such as, for example, $q = 0.5$ (dot-dashed line), 0.7 (dashed line) and 0.9 (solid line). In particular, these plots show how the curves associated with each n of a q-deformed HO are affected by the parameter $q \in (0,1)$. It is worth noting that such a phase distribution is a well-behaved function for both variables n and φ defined on a compact support, and it has a specular reflection (or symmetric behaviour) at the origin $\varphi = 0$.

an argument, figure 2 shows this particular phase distribution as a function of the angular variable $\varphi \in [-\pi,\pi]$ and different values of q, where the excitation degree n of the q-deformed HO is restricted into the closed interval [0, 4]—see figures 2(*a*)–(*e*).

(ii) The three-term recurrence relation (3) can be promptly adapted for $\{\mathscr{R}_n(z;q)\}_{n\in\mathbb{N}}$ as follows:

$$\mathscr{R}_{n+1}(z;q) = \left(\frac{q}{1-q^{n+1}}\right)^{\frac{1}{2}} \left\{(1+z)\mathscr{R}_n(z;q) - [q(1-q^n)]^{\frac{1}{2}} z\mathscr{R}_{n-1}(z;q)\right\}.$$

Furthermore, after some calculations based on the results obtained in [43] for the RS polynomials, it is easy to reach the additional relations

$$\mathscr{R}_n(z;q) - \mathscr{R}_n(qz;q) = [q(1-q^n)]^{\frac{1}{2}} z\mathscr{R}_{n-1}(z;q),$$
$$\mathscr{R}_n(qz;q) - q^n\mathscr{R}_n(z;q) = [q(1-q^n)]^{\frac{1}{2}} \mathscr{R}_{n-1}(qz;q),$$
$$\mathscr{R}_n(qz;q) - \mathscr{R}_n(q^2z;q) = [q(1-q^n)]^{\frac{1}{2}} qz\mathscr{R}_{n-1}(qz;q),$$
$$\mathscr{R}_n(z;q) - \mathscr{R}_n(q^2z;q) = q(1-q^n)z\mathscr{R}_n(z;q) + [q(1-q^n)]^{\frac{1}{2}}(1-qz)z\mathscr{R}_{n-1}(z;q).$$

In particular, these identities show how certain recurrence relations are modified by the scaling factors q and q^2.

(iii) This behaviour can also be verified for the weight function $\mathscr{M}(z;q)$, namely, scaling factors involving odd and even powers of the parameter q also modify equations (14) and (15)—or, in other words, they change the quasi-period of the Jacobi theta functions. In such cases, if one considers r an integer number, we obtain

(a) q^{2r}-case

$$\mathfrak{F}(q^{2r}z;q) = \mathrm{i}^r q^{-\frac{1}{2}r(r+\frac{1}{2})} z^{-\frac{r}{2}} \mathfrak{F}(z;q)$$
$$\mathfrak{G}(q^{2r}z;q) = (-\mathrm{i})^r q^{-\frac{1}{2}r(r+\frac{1}{2})} z^{-\frac{r}{2}} \mathfrak{G}(z;q)$$
$$\mathscr{M}(q^{2r}z;q) = (-1)^r q^{-r(r+\frac{1}{2})} z^{-r} \mathscr{M}(z;q)$$

and

(b) q^{2r+1}-case

$$\mathfrak{F}(q^{2r+1}z;q) = \mathrm{i}^{r+\frac{1}{2}} q^{-\frac{1}{2}(r+\frac{1}{2})(r+1)} z^{-\frac{1}{2}(r+\frac{1}{2})} \mathfrak{F}(z;q)$$
$$\mathfrak{G}(q^{2r+1}z;q) = (-\mathrm{i})^{r+\frac{1}{2}} q^{-\frac{1}{2}(r+\frac{1}{2})(r+1)} z^{-\frac{1}{2}(r+\frac{1}{2})} \mathfrak{G}(z;q)$$
$$\mathscr{M}(q^{2r+1}z;q) = (-1)^{r+\frac{1}{2}} q^{-(r+\frac{1}{2})(r+1)} z^{-(r+\frac{1}{2})} \mathscr{M}^*(z;q).$$

Note that scaling factors containing odd powers of q change the phase of the complex function $\mathscr{M}(z;q)$. Consequently, this result will bring some implications for the algebraic properties of $\Psi_n^{\mathrm{RS}}(z;q)$ and also will be responsible for the small change made in the usual definition of the Jackson's q-derivative [9, 10].

(iv) Adiga and coworkers [2, p 29] have established and proved several properties originated from Ramanujan's theorems on

$$f(a,b) := \sum_{\ell\in\mathbb{Z}} a^{\frac{1}{2}\ell(\ell+1)} b^{\frac{1}{2}\ell(\ell-1)},$$

where $|ab| < 1$ (we retain the original notation). Since a and b denote two complex variables in such a case, if one sets $a = q^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}\varphi}$ and $b = q^{\frac{1}{2}}\mathrm{e}^{-\mathrm{i}\varphi}$, it is immediate to verify that $f(a,b)$ coincides exactly with equation (11), i.e.,

$$f\left(q^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}\varphi}, q^{\frac{1}{2}}\mathrm{e}^{-\mathrm{i}\varphi}\right) = \left|\mathscr{M}\left(-q^{-\frac{1}{2}}\mathrm{e}^{\mathrm{i}\varphi};q\right)\right|^2 = |\mathscr{E}(\varphi;q)|^2 = \vartheta_3\left(\frac{\varphi}{2}\middle|q^{\frac{1}{2}}\right).$$

This particular connection allows us to increase the number of properties related to the Szegö measure (if one compares with those obtained in this work), as well as to derive new scaling relations.

3.2. Jackson's q-derivative

The next step consists in introducing a particular choice for Jackson's q-derivative through the action of a certain q-differential operator $\mathcal{D}_{q^2}$ on an arbitrary complex function $\phi(z;q)$ as follows[3]:

$$\mathcal{D}_{q^2}\phi(z;q) := \frac{\phi(z;q)-\phi(q^2z;q)}{z(1-q^2)}. \tag{21}$$

It is worth mentioning that some useful rules of q-differentiation, analogous to those verified for ordinary differentiation, can also be directly obtained in the context of q-calculus [44, 45]. Among them, let us focus on two basic rules which have an important role in our calculations, that is, the sum rule

$$\mathcal{D}_{q^2}\left[\phi_1(z;q)+\phi_2(z;q)\right] = \mathcal{D}_{q^2}\phi_1(z;q)+\mathcal{D}_{q^2}\phi_2(z;q) \tag{22}$$

and the q-version of the Leibniz rule

$$\begin{aligned}\mathcal{D}_{q^2}[\phi_1(z;q)\phi_2(z;q)] &= [\mathcal{D}_{q^2}\phi_1(z;q)]\phi_2(z;q)+\phi_1(q^2z;q)[\mathcal{D}_{q^2}\phi_2(z;q)]\\ &= [\mathcal{D}_{q^2}\phi_1(z;q)]\phi_2(q^2z;q)+\phi_1(z;q)[\mathcal{D}_{q^2}\phi_2(z;q)].\end{aligned} \tag{23}$$

As a first task, let us determine the action of the q-differential operator $\mathcal{D}_{q^2}$ on the polynomial $\mathscr{R}_n(z;q)$, that is,

$$\mathcal{D}_{q^2}\mathscr{R}_n(z;q) = \frac{q}{1+q}[n]_q\mathscr{R}_n(z;q)+\left(\frac{q}{1-q}\right)^{\frac{1}{2}}\frac{1-qz}{1+q}[n]_q^{1/2}\mathscr{R}_{n-1}(z;q) \tag{24}$$

where $[n]_q := \frac{1-q^n}{1-q}$ stands for the q-number [44, 45]. The second task then consists in finding out an expression for $\mathcal{D}_{q^2}\mathscr{M}(z;q)$ through the formal results obtained in the previous discussion about scaling factors,

$$\mathcal{D}_{q^2}\mathscr{M}(z;q) = \frac{1+q^{\frac{3}{2}}z}{q^{\frac{3}{2}}z^2(1-q^2)}\mathscr{M}(z;q). \tag{25}$$

Consequently, the action of $\mathcal{D}_{q^2}$ on $\Psi_n^{\mathrm{RS}}(z;q)$ can now be promptly obtained with the help of the q-Leibniz rule, i.e.,

$$\mathcal{D}_{q^2}\Psi_n^{\mathrm{RS}}(z;q) = [\mathcal{D}_{q^2}\mathscr{R}_n(z;q)]\mathscr{M}(q^2z;q)+\mathscr{R}_n(z;q)[\mathcal{D}_{q^2}\mathscr{M}(z;q)].$$

In this way, substituting equations (24) and (25) into this expression and after some minor adjustments in our calculations, we finally get the relations

$$\mathcal{D}_{q^2}\Psi_n^{\mathrm{RS}}(z;q) = \frac{1-qz\left(1-q^{\frac{1}{2}}-q^n\right)}{q^{\frac{3}{2}}z^2(1-q^2)}\Psi_n^{\mathrm{RS}}(z;q)-\left(\frac{[n]_q}{1-q}\right)^{\frac{1}{2}}\frac{1-qz}{qz(1+q)}\Psi_{n-1}^{\mathrm{RS}}(z;q)$$

and

$$\mathcal{D}_{q^2}\Psi_n^{\mathrm{RS}}(z;q) = -\frac{1-q\left(z+q^{\frac{1}{2}}+q^n\right)}{q^{\frac{3}{2}}z(1-q^2)}\Psi_n^{\mathrm{RS}}(z;q)+\left(\frac{[n+1]_q}{1-q}\right)^{\frac{1}{2}}\frac{1-qz}{q^2z^2(1+q)}\Psi_{n+1}^{\mathrm{RS}}(z;q)$$

for any $q\in(0,1)$ and $n\in\mathbb{N}$. Note that both identities not only depend on the degrees n and $n\mp 1$, but also preserve the phase of the orthogonalized RS functions—this fact being, in particular, a direct consequence of the scaling relations derived for $\mathscr{M}(z;q)$. The comparison of these results leads us to verify that $\Psi_n^{\mathrm{RS}}(z;q)$ satisfies a three-term recurrence relation exactly equal to that obtained for $\mathscr{R}_n(z;q)$.

[3] It is important to stress that there exist different versions of Jackson's q-derivative in the literature covering a wide range of applications in specific scenarios of mathematics and physics. For instance, Gelfand and coworkers [6] have proposed a two-parameter q-differential operator $\mathcal{D}_{r,s}$ whose action on $\phi(z;q)$ obeys the mathematical prescription

$$\mathcal{D}_{r,s}\phi(z;q) := \frac{\phi(rz;q)-\phi(sz;q)}{z(r-s)}.$$

Note that $\mathcal{D}_{q^2}$ represents a particular case of $\mathcal{D}_{r,s}$ since $\mathcal{D}_{q^2}\equiv\mathcal{D}_{1,q^2}$.

3.3. Lowering and raising operators

As a last step in our calculations, let us now construct the q-differential representations of the lowering and raising operators associated with $\left\{\Psi_n^{\mathrm{RS}}(z;q)\right\}_{n\in\mathbb{N}}$, as well as the respective representation for the q-deformed number operator. For this intent, both the results obtained for $\mathcal{D}_{q^2}\Psi_n^{\mathrm{RS}}(z;q)$ are taken into account in this process, asserting that

$$\widehat{L}_n(z;q) := \frac{1 - qz\big(1 - q^{\frac{1}{2}} - q^n\big) - q^{\frac{3}{2}}z^2(1-q^2)\mathcal{D}_{q^2}}{[q(1-q)]^{\frac{1}{2}}(1-qz)z} \tag{26}$$

and

$$\widehat{R}_n(z;q) := \frac{q^{\frac{1}{2}}z\big[1 - q\big(z + q^{\frac{1}{2}} + q^n\big) + q^{\frac{3}{2}}z(1-q^2)\mathcal{D}_{q^2}\big]}{(1-q)^{\frac{1}{2}}(1-qz)} \tag{27}$$

represent—within the range of possibilities related to the definition of Jackson's q-derivative applied to the problem under scrutiny—two legitimate q-differential representations of the lowering ($\mathbf{B}$) and raising ($\mathbf{B}^\dagger$) operators, respectively, whose actions on the RS functions result in[4]

$$\mathbf{B}\Psi_n^{\mathrm{RS}}(z;q) \equiv \widehat{L}_n(z;q)\Psi_n^{\mathrm{RS}}(z;q) = [n]_q^{1/2}\Psi_{n-1}^{\mathrm{RS}}(z;q),$$
$$\mathbf{B}^\dagger\Psi_n^{\mathrm{RS}}(z;q) \equiv \widehat{R}_n(z;q)\Psi_n^{\mathrm{RS}}(z;q) = [n+1]_q^{1/2}\Psi_{n+1}^{\mathrm{RS}}(z;q).$$

Moreover, if one defines

$$\widehat{N}_n(z;q) := \widehat{R}_n(z;q)\widehat{L}_n(z;q) \tag{28}$$

as the number operator $\mathbf{N}_q$ in its q-differential form, it is immediate to show that

$$\mathbf{N}_q\Psi_n^{\mathrm{RS}}(z;q) \equiv \widehat{N}_n(z;q)\Psi_n^{\mathrm{RS}}(z;q) = [n]_q\Psi_n^{\mathrm{RS}}(z;q).$$

Such q-differential representations can be considered as a particular realization (within a wide class of representations with different applications in mathematics and physics) of the IACK algebra [20] here characterized by the commutation relations

- q-commutation relation $\big([\mathbf{X},\mathbf{Y}]_q \equiv \mathbf{XY} - q\mathbf{YX}\big)$

$$[\mathbf{B},\mathbf{B}^\dagger]_q = \mathbf{1}, \quad [\mathbf{B},\mathbf{N}_q]_q = \mathbf{B}, \quad [\mathbf{N}_q,\mathbf{B}^\dagger]_q = \mathbf{B}^\dagger, \tag{29}$$

- standard commutation relation ($[\mathbf{X},\mathbf{Y}] \equiv \mathbf{XY} - \mathbf{YX}$)

$$[\mathbf{B},\mathbf{B}^\dagger] = \mathbf{1} - (1-q)\mathbf{N}_q, \quad [\mathbf{N}_q,\mathbf{B}] = -[\mathbf{1} - (1-q)\mathbf{N}_q]\mathbf{B},$$
$$[\mathbf{N}_q,\mathbf{B}^\dagger] = \mathbf{B}^\dagger[\mathbf{1} - (1-q)\mathbf{N}_q]. \tag{30}$$

Next, we will discuss certain relevant points related to the results previously obtained in this paragraph.

Our first comment refers to the q-differential forms determined for the lowering, raising and number operators and their dependences on the degree n, this fact being interpreted as a direct consequence of the definition employed in this work for the RS functions and its inherent properties—indeed, the expressions for $\mathcal{D}_{q^2}\Psi_n^{\mathrm{RS}}(z;q)$ already bring such dependence. Besides, the energy eigenvalues [20, 29]

$$E_n = \frac{2 - (1+q)q^n}{1-q}E_0 \qquad (E_0 = \hbar\omega_0/2)$$

[4] Although the q-differential representations $\widehat{L}_n(z;q)$ and $\widehat{R}_n(z;q)$ present an explicit dependence on the discrete variable n (which certainly represents a possible drawback in our algebraic approach), it is worth stressing that the matrix elements of the raising and lowering operators can always be evaluated via the RS-functions representation, and they are sufficient to properly and uniquely determine such operators.

of this particular q-deformed HO present a nonlinear dependence on n and its energy spectrum is not equally spaced, namely

$$\Delta_n := \frac{E_{n+1} - E_n}{E_0} = (1+q)q^n.$$

In principle, such arguments should be sufficient to explain qualitatively the functional forms of equations (26) and (27).

The second comment is related to the commutation relations (29) and (30), where it is clear that both relations are equivalent only in the contraction limit $q \to 1^-$ (in this limit, we recover the Heisenberg–Weyl algebra $\mathfrak{h}_4$); furthermore, analogous commutation relations were also obtained in [40] for the RS polynomials. As a last remark, let us mention that Floreanini and Vinet [37] have developed a quantum-algebraic framework for a finite set of q-special functions, where the realizations of the underlying algebras are given in terms of operators acting on vector spaces of complex functions. In this sense, the results discussed here may represent an important contribution to that framework since they allow us to open a new promising chapter on the angular representations in quantum mechanics.

4. Coherent states

In accordance with the algebraic approach developed until now, let us construct in this section a class of Barut–Girardello coherent states [51] related to the RS functions which are eigenstates of the lowering operator $\mathbf{B}$ with eigenvalues $\mu \in \mathbb{C}$, that is,

$$\mathbf{B}\mathscr{F}_\mu(z;q) = \mu\mathscr{F}_\mu(z;q). \tag{31}$$

Within this context, the complex function $\mathscr{F}_\mu(z;q)$ can be properly expanded in terms of the complete set $\left\{\Psi_n^{\mathrm{RS}}(z;q)\right\}_{n\in\mathbb{N}}$ as follows [23]:

$$\mathscr{F}_\mu(z;q) = \sum_{n\in\mathbb{N}} \mathscr{C}_n(\mu;q)\Psi_n^{\mathrm{RS}}(z;q). \tag{32}$$

The coefficients $\{\mathscr{C}_n(\mu;q)\}_{n\in\mathbb{N}^*}$ are then determined by means of equation (31), while the coefficient $\mathscr{C}_0(\mu;q)$ results from the orthogonality relation (16). Consequently, such a mathematical procedure leads us to obtain

$$\mathscr{C}_n(\mu;q) = e_q^{-1/2}((1-q)|\mu|^2)\frac{\mu^n}{\sqrt{[n]_q!}},$$

where $e_q(x) := (x;q)_\infty^{-1}$ defines a q-exponential function which converges absolutely for $q \in (0,1)$ and $|x| < 1$ [9, p 9]. Therefore, if one substitutes these coefficients on the right-hand side of equation (32) and takes into account the identity

$$\sum_{n\in\mathbb{N}} \mathscr{H}_n(z;q)\frac{\left[\sqrt{q(1-q)}\mu\right]^n}{(q;q)_n} = \left(\sqrt{q(1-q)}\mu, \sqrt{q(1-q)}\mu z;q\right)_\infty^{-1},$$

which, as has been pointed out in section 2, can be readily attained via equation (2), we finally obtain the entire function

$$\mathscr{F}_\mu(z;q) = \frac{e_q^{-1/2}((1-q)|\mu|^2)\mathscr{M}(z;q)}{\sqrt{2\pi}\left(\sqrt{q(1-q)}\mu, \sqrt{q(1-q)}\mu z;q\right)_\infty}. \tag{33}$$

Note that equations (32) and (33) represent two different forms of expressing $\mathscr{F}_\mu(z;q)$, and that $|\mathscr{C}_n(\mu;q)|^2$ is associated with the excitation probability distribution for the q-deformed coherent state. In the following, let us present at least two inherent algebraic properties of these particular coherent states.

The first property is a direct consequence of expansion (33) and corresponds to the formula for the scalar product

$$\int_{-\pi}^{\pi} \mathscr{F}_{\nu}^{*}\big(-q^{-\frac{1}{2}}\mathrm{e}^{\mathrm{i}\varphi}; q\big)\mathscr{F}_{\mu}\big(-q^{-\frac{1}{2}}\mathrm{e}^{\mathrm{i}\varphi}; q\big)\,\mathrm{d}\varphi = \frac{e_q((1-q)\mu\nu^{*})}{[e_q((1-q)|\mu|^2)e_q((1-q)|\nu|^2)]^{\frac{1}{2}}}, \tag{34}$$

from which we can infer the inequality

$$0 < \frac{|e_q((1-q)\mu\nu^{*})|^2}{e_q((1-q)|\mu|^2)e_q((1-q)|\nu|^2)} \leqslant 1.$$

Note that this overlap probability is equal to one for $\nu = \mu$ (normalizability condition of the scalar product) and falls to zero when $|\mu - \nu|^2$ becomes large in the limit $q \to 1^{-}$; in other words, the coherent states $\{\mathscr{F}_{\mu}(z; q)\}_{\mu\in\mathbb{C}}$ are not orthogonal. Gray and Nelson [24] obtained an analogous mathematical result for the scalar product of coherent states associated with the q-deformed HO, whose commutation relations are characterized by $\mathbf{A}\mathbf{A}^{\dagger} - q^{-1/2}\mathbf{A}^{\dagger}\mathbf{A} = q^{-\mathbf{N}/2}$, $[\mathbf{N}, \mathbf{A}^{\dagger}] = \mathbf{A}^{\dagger}$, and $[\mathbf{N}, \mathbf{A}] = -\mathbf{A}$. Such a result is not a mere coincidence since both approaches can be considered as particular cases of the generalized q-deformed Heisenberg–Weyl algebra $\mathrm{U}_q^{(\alpha,\beta,\gamma)}(\mathfrak{h}_4)$ [29].

The next property assures the implicit resolution of unity (or completeness relation) for the coherent states defined by means of the entire function $\mathscr{F}_{\mu}(z; q)$, namely

$$\int_{\mathbb{D}_q} \mathscr{F}_{\mu}^{*}(w; q)\mathscr{F}_{\mu}(z; q) d^2\sigma_q(\mu) = \lim_{\varepsilon\to 1^{-}} K_{\varepsilon}(w, z; q), \tag{35}$$

where

$$\mathbb{D}_q = \left\{\mu = |\mu|\mathrm{e}^{\mathrm{i}\theta} : |\mu|^2 \in \left[0, \frac{1}{q-1}\right) \quad \text{and} \quad \theta \in [0, 2\pi)\right\}$$

corresponds to the integration domain in the complex plane,

$$\mathrm{d}^2\sigma_q(\mu) = \frac{e_q((1-q)|\mu|^2)}{e_q((1-q)q|\mu|^2)} \frac{\mathrm{d}_q(|\mu|^2)\,\mathrm{d}\theta}{2\pi} \tag{36}$$

its respective measure [26], and $K_{\varepsilon}(w, z; q)$ represents the bilinear kernel (17). To demonstrate this specific identity, we first substitute expansion (32) into the left-hand side of equation (35) and then carry out, subsequently, the integration over the angle variable θ. This common procedure permits us to reduce the integration over the domain $\mathbb{D}_q$ in the Jackson q-integral [23, 45]

$$I_q(n) = \int_0^{\frac{1}{1-q}} \frac{|\mu|^{2n}}{e_q((1-q)q|\mu|^2)} d_q(|\mu|^2) = \frac{1}{(1-q)^n} \sum_{k\in\mathbb{N}} \frac{q^{k(n+1)}}{e_q(q^{k+1})}.$$

As an intermediate stage in our proof, let us now consider the result derived in [9] for the q-exponential function $e_q(q^{k+1})$, i.e., $e_q^{-1}(q^{k+1}) = (q; q)_{\infty}/(q; q)_k$. Consequently, substituting this result into $I_q(n)$, it is then immediate to show that

$$I_q(n) = \frac{(q; q)_{\infty}}{(1-q)^n} \sum_{k\in\mathbb{N}} \frac{q^{k(n+1)}}{(q; q)_k} = \frac{(q; q)_{\infty}}{(1-q)^n (q^{n+1}; q)_{\infty}} = \frac{(q; q)_n}{(1-q)^n} = [n]_q!.$$

So, after some minor adjustments in our calculations, the right-hand side of equation (35) can be promptly reached.

To finish this section, let us discuss three important points raised by the identity (35). The first point corresponds to the resolution of unity for the coherent states which is implicitly demonstrated in our evaluations. In its turn, combining this result with the first property, we can conclude that equation (31) produces a special set of overcomplete q-deformed coherent states. The second point is entirely related to the term appearing on the right-hand side of equation (35): it reflects the normalization condition for the complex representations used in this work. Finally, the last point is associated with the evidence that $\mathrm{d}^2\sigma_q(\mu)$ may not be unique as some studies that appeared in the literature [27, 28] indicate. In fact, different integration measures can produce distinct integrals $\mathcal{I}_q(n)$ whose integrands, in their turn, are related to solutions of Stieltjes and Hausdorff moment problems [55, 56].

5. Phase states

Previously, in section 3, we have established the q-differential forms for the lowering, raising and number operators whose respective actions on the RS functions resemble those usually obtained for the annihilation, creation and number operators when acting on the wavefunctions $\left\{\Psi_n^{\mathrm{HO}}(x;\alpha)\right\}_{n\in\mathbb{N}}$ associated with the usual harmonic oscillator. This particular analogy leads us to investigate the possibility of constructing a polar decomposition for the lowering (raising) operator $\mathbf{B}$ $(\mathbf{B}^\dagger)$ through the equivalence relation $\mathbf{B} \equiv [\mathbf{N}+\mathbf{1}]_q^{1/2}\boldsymbol{\mathcal{E}}_-$ $(\mathbf{B}^\dagger \equiv \boldsymbol{\mathcal{E}}_+[\mathbf{N}+\mathbf{1}]_q^{1/2})$, where $\mathbf{N}$ denotes the standard number operator[5]. Moreover, $\boldsymbol{\mathcal{E}}_\mp$ represent two 'exponential' operators which will be explored adequately in this section. It is important to stress that the underlying problems of this specific decomposition and their possible solutions with convenient inherent mathematical properties were extensively discussed in the literature [49, 50]. Here, our focus will be the construction of orthogonal eigenstates related to the q-deformed cosine and sine operators, in analogy with the results discussed in [46, 47], namely

$$\boldsymbol{\mathcal{C}} := \frac{1}{2}\left(\boldsymbol{\mathcal{E}}_- + \boldsymbol{\mathcal{E}}_+\right) \quad \text{and} \quad \boldsymbol{\mathcal{S}} := \frac{1}{2\mathrm{i}}\left(\boldsymbol{\mathcal{E}}_- - \boldsymbol{\mathcal{E}}_+\right), \tag{37}$$

as well as the calculation of certain mean values pertinent to $\boldsymbol{\mathcal{C}}$ and $\boldsymbol{\mathcal{S}}$ which allow us to determine some intrinsic properties of the coherent states described in section 4.

Initially, let us assume that $\boldsymbol{\mathcal{E}}_\mp$ acting on the RS functions $\Psi_n^{\mathrm{RS}}(z;q)$ results in the identity $\boldsymbol{\mathcal{E}}_\mp\Psi_n^{\mathrm{RS}}(z;q) \equiv \Psi_{n\mp 1}^{\mathrm{RS}}(z;q)$, i.e., its action decreases/increases the excitation degree n of the q-deformed HO by one [46]. The next step then consists in solving the eigenvalue equation

$$\boldsymbol{\mathcal{C}}\mathscr{X}_\gamma(z;q) = \cos(\gamma)\mathscr{X}_\gamma(z;q) \tag{38}$$

following the mathematical recipe described in [47]. Note that $\{\mathscr{X}_\gamma(z;q)\}_{\gamma\in[0,\pi]}$ can be expanded in terms of the complete set $\left\{\Psi_n^{\mathrm{RS}}(z;q)\right\}_{n\in\mathbb{N}}$, and their respective coefficients determined in order to obey the eigenvalue equation (38). Thus, after some nontrivial algebra, we obtain

$$\mathscr{X}_\gamma(z;q) = \sqrt{\frac{2}{\pi}}\sum_{n\in\mathbb{N}}\sin[(n+1)\gamma]\Psi_n^{\mathrm{RS}}(z;q), \tag{39}$$

which satisfies not only the orthogonality relation

$$\int_{-\pi}^{\pi}\mathscr{X}_{\gamma'}^*\left(-q^{-\frac{1}{2}}\mathrm{e}^{\mathrm{i}\varphi};q\right)\mathscr{X}_\gamma\left(-q^{-\frac{1}{2}}\mathrm{e}^{\mathrm{i}\varphi};q\right)\,\mathrm{d}\varphi = \delta(\gamma'-\gamma) \tag{40}$$

but also the resolution of unity implicitly expressed as

$$\int_0^{\pi}\mathscr{X}_\gamma^*(w;q)\mathscr{X}_\gamma(z;q)\,\mathrm{d}\gamma = \lim_{\varepsilon\to 1^-}K_\varepsilon(w,z;q). \tag{41}$$

[5] It is important to mention that $\mathbf{N}_q \equiv \mathbf{B}^\dagger\mathbf{B}$ coincides with the standard number operator $\mathbf{N}$ only in the limit $q \to 1^-$ [29]. In this case, the operator $\mathbf{N}$ is subjected to the commutation relations $[\mathbf{N},\mathbf{B}] = -\mathbf{B}$ and $[\mathbf{N},\mathbf{B}^\dagger] = \mathbf{B}^\dagger$, which differ, in their turn, from those obtained in equation (30) for $\mathbf{N}_q$.

Now, let us mention an important feature inherent to expansion (39): it vanishes at the points $\gamma = 0$ or π, and this fact implies the non-existence of singularities at these points since there are no states related to them [47].

The construction process of the eigenfunctions related to $\mathcal{S}$ follows along similar lines, namely, they are solutions of the eigenvalue equation

$$\mathcal{S}\mathscr{Y}_\gamma(z;q) = \sin(\gamma)\mathscr{Y}_\gamma(z;q) \tag{42}$$

whose expansion in terms of the RS functions obeys the equation

$$\mathscr{Y}_\gamma(z;q) = \mathrm{i}\sqrt{\frac{2}{\pi}}\sum_{n\in\mathbb{N}} \mathrm{e}^{\mathrm{i}(n+1)\frac{\pi}{2}} \sin\left[(n+1)\left(\gamma - \frac{\pi}{2}\right)\right]\Psi_n^{\mathrm{RS}}(z;q). \tag{43}$$

In analogy with properties (40) and (41) we also find the following relations:

$$\int_{-\pi}^{\pi} \mathscr{Y}_{\gamma'}^{*}\left(-q^{-\frac{1}{2}}\mathrm{e}^{\mathrm{i}\varphi};q\right)\mathscr{Y}_\gamma\left(-q^{-\frac{1}{2}}\mathrm{e}^{\mathrm{i}\varphi};q\right)\,\mathrm{d}\varphi = \delta(\gamma'-\gamma), \tag{44}$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathscr{Y}_\gamma^{*}(w;q)\mathscr{Y}_\gamma(z;q)\,\mathrm{d}\gamma = \lim_{\varepsilon\to 1^-} K_\varepsilon(w,z;q). \tag{45}$$

It is worth stressing that the eigenfunctions here obtained for the Hermitian operators $\mathcal{C}$ and $\mathcal{S}$ depend strongly on the action of $\mathcal{E}_\mp$ upon the complete set $\left\{\Psi_n^{\mathrm{RS}}(z;q)\right\}_{n\in\mathbb{N}}$. In other words, different polar decompositions for the lowering operator (as well as distinct assumptions about the action of $\mathcal{E}_\mp$) lead us to derive different expressions for $\mathscr{X}_\gamma(z;q)$ and $\mathscr{Y}_\gamma(z;q)$. This fact was properly explored by Bergou and Englert [48] within the Wigner function context and its asymptotic form for a quantum operator, where, in particular, the authors showed how to construct different phase operators related to the usual HO.

6. Applications

6.1. Mean values

In the following, we derive a set of closed-form expressions for certain moments related to the cosine and sine operators evaluated via coherent states $\{\mathscr{F}_\mu(z;q)\}_{\mu\in\mathbb{C}}$. Our main task then consists in computing initially some specific mean values involving $\mathcal{C}$ and $\mathcal{S}$ by means of the auxiliary relation

$$\langle\mathbf{O}\rangle_\mu = \int_{-\pi}^{\pi} \mathscr{F}_\mu^{*}\left(-q^{-\frac{1}{2}}\mathrm{e}^{\mathrm{i}\varphi};q\right)\mathbf{O}\mathscr{F}_\mu\left(-q^{-\frac{1}{2}}\mathrm{e}^{\mathrm{i}\varphi};q\right)\,\mathrm{d}\varphi. \tag{46}$$

It is important to mention that the action of such operators on the coherent states is not trivial, namely it presents peculiarities inherent to the nonunitarity of $\mathcal{E}_-$. So, after lengthy calculations, we obtain for $\mu = |\mu|\mathrm{e}^{\mathrm{i}\theta}$ the exact results [25]

$$\langle\mathcal{C}\rangle_\mu = |\mu| e_q^{-1}((1-q)|\mu|^2)\mathcal{M}_q(|\mu|^2)\cos(\theta),$$
$$\langle\mathcal{S}\rangle_\mu = |\mu| e_q^{-1}((1-q)|\mu|^2)\mathcal{M}_q(|\mu|^2)\sin(\theta),$$
$$\langle\mathcal{C}^2\rangle_\mu = \frac{1}{2} - \frac{1}{4}e_q^{-1}((1-q)|\mu|^2) + \frac{1}{2}|\mu|^2\, e_q^{-1}((1-q)|\mu|^2)\mathcal{N}_q(|\mu|^2)\cos(2\theta),$$
$$\langle\mathcal{S}^2\rangle_\mu = \frac{1}{2} - \frac{1}{4}e_q^{-1}((1-q)|\mu|^2) - \frac{1}{2}|\mu|^2\, e_q^{-1}((1-q)|\mu|^2)\mathcal{N}_q(|\mu|^2)\cos(2\theta),$$
$$\langle\mathcal{C}^2+\mathcal{S}^2\rangle_\mu = 1 - \frac{1}{2}e_q^{-1}((1-q)|\mu|^2),$$

$$\langle \mathcal{C}^2 - \mathcal{S}^2 \rangle_\mu = |\mu|^2 \, e_q^{-1}((1-q)|\mu|^2)\mathcal{N}_q(|\mu|^2)\cos(2\theta),$$
$$\langle \mathcal{C}\mathcal{S} + \mathcal{S}\mathcal{C} \rangle_\mu = |\mu|^2 \, e_q^{-1}((1-q)|\mu|^2)\mathcal{N}_q(|\mu|^2)\sin(2\theta),$$
$$\langle \mathcal{C}\mathcal{S} - \mathcal{S}\mathcal{C} \rangle_\mu = \frac{\mathrm{i}}{2} \, e_q^{-1}((1-q)|\mu|^2),$$

where the functions $\mathcal{M}_q(|\mu|^2)$ and $\mathcal{N}_q(|\mu|^2)$ are here defined by the power series

$$\mathcal{M}_q(|\mu|^2) = \sum_{n\in\mathbb{N}} \frac{|\mu|^{2n}}{[n]_q!([n+1]_q)^{\frac{1}{2}}} \quad \text{and} \quad \mathcal{N}_q(|\mu|^2) = \sum_{n\in\mathbb{N}} \frac{|\mu|^{2n}}{[n]_q!([n+2]_q[n+1]_q)^{\frac{1}{2}}}.$$

Such mean values can be interpreted as the q-deformed version of those obtained by Carruthers and Nieto [47]. Furthermore, for $q \to 1^-$ and $|\mu|^2 \gg 1$, these exact results reach the asymptotic limits (e.g., see Lynch [49, p 378] for further discussion)

$$\langle \mathcal{C} \rangle_\mu \approx \cos(\theta), \quad \langle \mathcal{S} \rangle_\mu \approx \sin(\theta), \quad \langle \mathcal{C}^2 \rangle_\mu \approx \cos^2(\theta), \quad \langle \mathcal{S}^2 \rangle_\mu \approx \sin^2(\theta), \quad \langle \mathcal{C}^2 + \mathcal{S}^2 \rangle_\mu \approx 1,$$
$$\langle \mathcal{C}^2 - \mathcal{S}^2 \rangle_\mu \approx \cos(2\theta), \quad \langle \mathcal{C}\mathcal{S} + \mathcal{S}\mathcal{C} \rangle_\mu \approx \sin(2\theta), \quad \langle \mathcal{C}\mathcal{S} - \mathcal{S}\mathcal{C} \rangle_\mu \approx 0.$$

Figure 3 shows the plots of the first six mean values versus $|\mu|^2 \in [0, 5]$ with $\theta = \frac{\pi}{3}$ fixed, and different values of q such that $(1-q)|\mu|^2 < 1$ is satisfied. The asymptotic limits observed in the numerical calculations are in agreement with those predicted theoretically, and this fact will be our reference in the study of Robertson–Schrödinger uncertainty relations for the cosine and sine operators.

6.2. Symmetrical uncertainty relation

As a last topic of interest, let us now study qualitatively the Robertson–Schrödinger uncertainty relation [57]

$$\mathscr{U}_{\mathcal{C}\mathcal{S}} := \mathscr{V}_{\mathcal{C}}\mathscr{V}_{\mathcal{S}} - (\mathscr{V}_{\mathcal{C}\mathcal{S}})^2 \geqslant \frac{1}{4}|\langle[\mathcal{C}, \mathcal{S}]\rangle_\mu|^2 \tag{47}$$

through the mean values established for the coherent states, where

$$\mathscr{V}_{\mathcal{C}} \equiv \langle \mathcal{C}^2 \rangle_\mu - \langle \mathcal{C} \rangle_\mu^2, \quad \mathscr{V}_{\mathcal{S}} \equiv \langle \mathcal{S}^2 \rangle_\mu - \langle \mathcal{S} \rangle_\mu^2, \quad \text{and} \quad \mathscr{V}_{\mathcal{C}\mathcal{S}} \equiv \left\langle \frac{1}{2}\{\mathcal{C}, \mathcal{S}\} \right\rangle_\mu - \langle \mathcal{C} \rangle_\mu \langle \mathcal{S} \rangle_\mu$$

represent the variances related to the cosine and sine operators. In addition, the terms $\langle[\mathcal{C}, \mathcal{S}]\rangle_\mu$ and $\langle\{\mathcal{C}, \mathcal{S}\}\rangle_\mu$ correspond to the commutation and anticommutation relation mean values, respectively. Thus, after a straightforward calculation, we find that $\mathscr{U}_{\mathcal{C}\mathcal{S}}$ does not depend on the angle variable θ, but only on the parameters q and $|\mu|$. Indeed, using the results previously obtained in the last section, if one denotes

$$\mathfrak{a} = \frac{1}{2} - \frac{1}{4} e_q^{-1}((1-q)|\mu|^2),$$
$$\mathfrak{b} = \frac{1}{2}|\mu|^2 e_q^{-1}((1-q)|\mu|^2)\mathcal{N}_q(|\mu|^2),$$
$$\mathfrak{c} = |\mu| e_q^{-1}((1-q)|\mu|^2)\mathcal{M}_q(|\mu|^2),$$

the left-hand side of equation (47) can be properly written as $\mathscr{U}_{\mathcal{C}\mathcal{S}} = (\mathfrak{a} - \mathfrak{b})(\mathfrak{a} + \mathfrak{b} - \mathfrak{c}^2)$. It is worth emphasizing that our numerical evaluations corroborate the inequality $\mathscr{U}_{\mathcal{C}\mathcal{S}} \geqslant \frac{1}{4}|\langle[\mathcal{C}, \mathcal{S}]\rangle_\mu|^2$. Moreover, for $q \to 1^-$ and $|\mu|^2 \gg 1$, we obtain $|\langle[\mathcal{C}, \mathcal{S}]\rangle_\mu|^2 \to 0$, which implies that $\mathcal{C}$ and $\mathcal{S}$ can be considered as commutative variables [47].

Next, let us derive a symmetrical relation which involves a particular combination of the number-cosine and number-sine Robertson–Schrödinger uncertainty relations, namely,

$$\mathscr{V}_{\mathrm{N}}\mathscr{V}_{\mathcal{C}} - (\mathscr{V}_{\mathrm{N}\mathcal{C}})^2 \geqslant \frac{1}{4}|\langle[\mathrm{N}, \mathcal{C}]\rangle_\mu|^2 = \frac{1}{4}\langle \mathcal{S} \rangle_\mu^2, \tag{48}$$

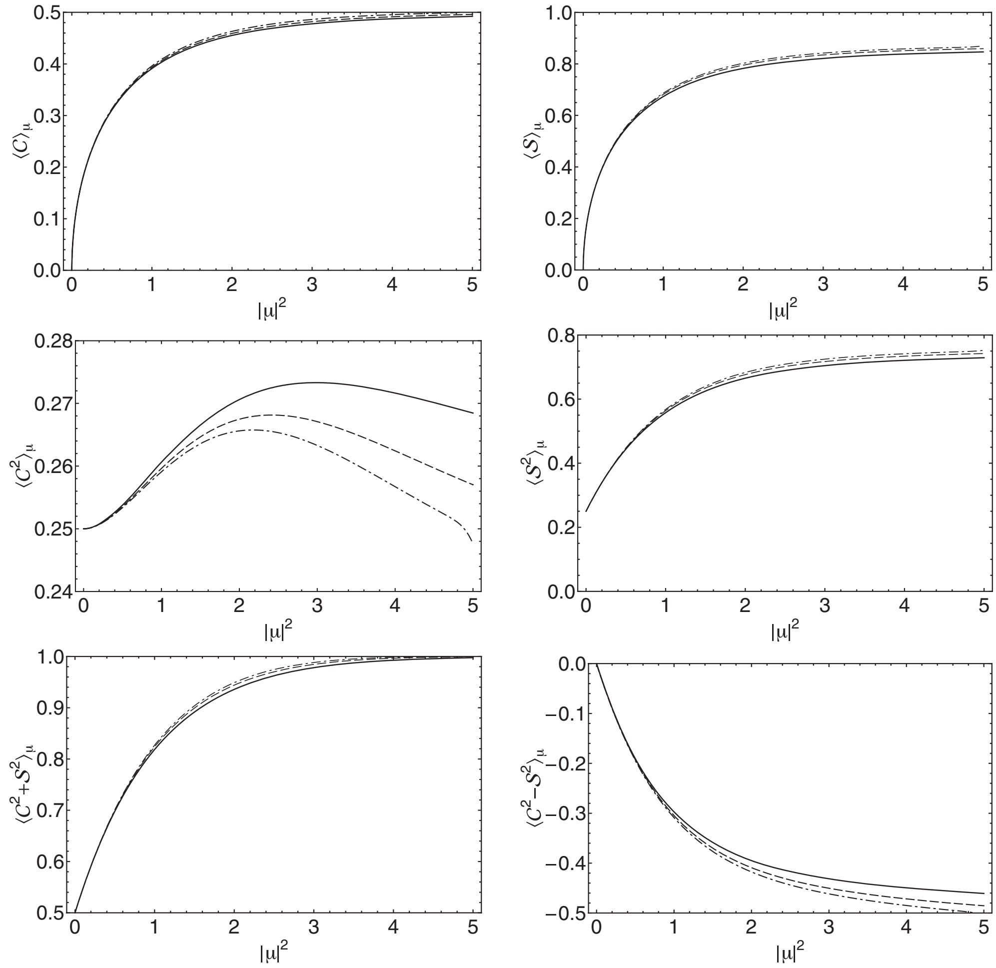

Figure 3. Plots of first- and second-order moments involving the cosine and sine operators as a function of $0 \leqslant |\mu|^2 \leqslant 5$ with $\theta = \frac{\pi}{3}$ fixed, and different values of q. In such examples, the dot-dashed, dashed and solid lines correspond, respectively, to the values of $q = 0.8, 0.85$ and 0.9. Note that the distinct asymptotic values reached in each case exhibit a strong dependence on the parameters q and $|\mu|^2$, this fact being associated with the convergence criterion $(1 - q)|\mu|^2 < 1$ adopted for the q-exponential function.

$$\mathscr{V}_{\mathrm{N}}\mathscr{V}_{\mathcal{S}} - (\mathscr{V}_{\mathrm{N}\mathcal{S}})^2 \geqslant \frac{1}{4}|\langle[\mathbf{N}, \boldsymbol{\mathcal{S}}]\rangle_\mu|^2 = \frac{1}{4}\langle\boldsymbol{\mathcal{C}}\rangle_\mu^2 \tag{49}$$

where $\mathscr{V}_{\mathrm{N}} \equiv \langle\mathbf{N}^2\rangle_\mu - \langle\mathbf{N}\rangle_\mu^2$ represents the variance related to the nondeformed number operator, with

$$\mathscr{V}_{\mathrm{N}\mathcal{C}} \equiv \left\langle\frac{1}{2}\{\mathbf{N}, \boldsymbol{\mathcal{C}}\}\right\rangle_\mu - \langle\mathbf{N}\rangle_\mu\langle\boldsymbol{\mathcal{C}}\rangle_\mu \quad \text{and} \quad \mathscr{V}_{\mathrm{N}\mathcal{S}} \equiv \left\langle\frac{1}{2}\{\mathbf{N}, \boldsymbol{\mathcal{S}}\}\right\rangle_\mu - \langle\mathbf{N}\rangle_\mu\langle\boldsymbol{\mathcal{S}}\rangle_\mu$$

being the corresponding covariances associated with the number-cosine and number-sine operators. Our next step consists in adding (48) and (49) with the aim of obtaining the symmetrical relation

$$\mathscr{U}_{\mathrm{sym}} \equiv \frac{\mathscr{V}_{\mathrm{N}}(\mathscr{V}_{\mathcal{C}} + \mathscr{V}_{\mathcal{S}}) - \left[(\mathscr{V}_{\mathrm{N}\mathcal{C}})^2 + (\mathscr{V}_{\mathrm{N}\mathcal{S}})^2\right]}{\langle\boldsymbol{\mathcal{S}}\rangle_\mu^2 + \langle\boldsymbol{\mathcal{C}}\rangle_\mu^2} \geqslant \frac{1}{4} \tag{50}$$

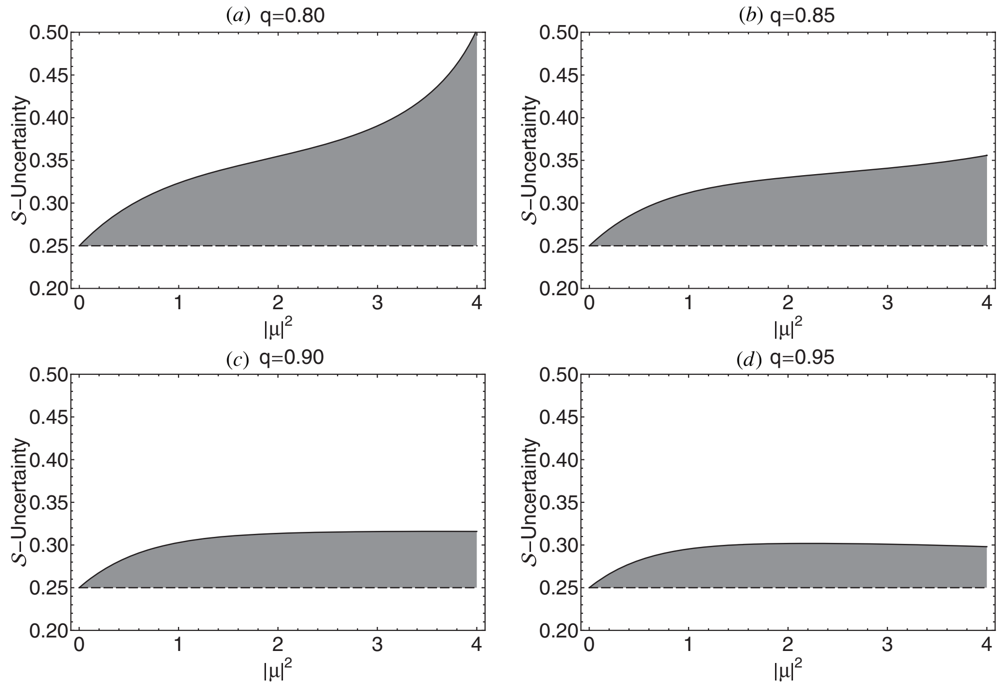

Figure 4. Plots of $\mathscr{U}_{\mathrm{sym}}$ (solid line) as a function of $|\mu|^2 \in [0, 4]$ for different values of q. In all cases, the dashed line corresponds to the constant value $\frac{1}{4}$, which reflects the pattern behaviour observed for the nondeformed coherent states. Note that for values of q close to 1^-, the filled space (grey colour) between the curves diminishes, and when $|\mu|^2 \gg 1$ we obtain the asymptotic limit $\mathscr{U}_{\mathrm{sym}} \to \frac{1}{4}$.

between $\mathcal{C}$ and $\mathcal{S}$, which does not depend on the angle variable θ. Indeed, in order to prove this assertion it is sufficient to calculate the additional mean values

$$\langle \mathbf{N}^k \rangle_\mu = e_q^{-1}((1-q)|\mu|^2) \sum_{n \in \mathbb{N}} \frac{n^k |\mu|^{2n}}{[n]_q!} \quad (k \geqslant 0),$$

$$\left\langle \frac{1}{2}\{\mathbf{N}, \mathcal{C}\} \right\rangle_\mu = \frac{|\mu|}{2} e_q^{-1}((1-q)|\mu|^2) \mathcal{L}_q(|\mu|^2) \cos(\theta),$$

$$\left\langle \frac{1}{2}\{\mathbf{N}, \mathcal{S}\} \right\rangle_\mu = \frac{|\mu|}{2} e_q^{-1}((1-q)|\mu|^2) \mathcal{L}_q(|\mu|^2) \sin(\theta),$$

with $\mathcal{L}_q(|\mu|^2)$ given by the power series

$$\mathcal{L}_q(|\mu|^2) = \sum_{n \in \mathbb{N}} \frac{(2n+1)|\mu|^{2n}}{[n]_q!([n+1]_q)^{\frac{1}{2}}}.$$

Figure 4 illustrates the symmetrical relation (50) as a function of $|\mu|^2 \in [0, 4]$ for (*a*) $q = 0.80$, (*b*) $q = 0.85$, (*c*) $q = 0.90$ and (*d*) $q = 0.95$. In all these pictures, the solid line corresponds to the left-hand side of the inequality (50), while the right-hand side is represented by the dashed line. Now, let us say a few words about the filled space (grey colour) between the curves: since the convergence criterion $0 \leqslant (1-q)|\mu|^2 < 1$ restricts the domain of the variables q and $|\mu|^2$, we can verify that the area related to this space decreases for values of q near to 1^- and $0 < |\mu|^2 \leqslant 4$—see figures 4(*a*) and (*b*). A plausible explanation of this fact leads us to the q-deformed coherent states—here represented by the eigenfunctions $\{\mathscr{F}_{\mu}(z;q)\}_{\mu\in\mathbb{C}}$—and their inherent properties, i.e., these particular states are not minimum uncertainty states (excepting the vacuum state $|\mu| = 0$), which justifies, in principle, the appearance of this specific region between the curves and its possible variation. As a last comment, let us mention that $\mathscr{V}_{\mathrm{N}} \neq \langle \mathbf{N} \rangle_{\mu}$ in such a case, which implies that $\{|\mathscr{C}_n(\mu;q)|^2\}_{n\in\mathbb{N}}$ are not Poisson distributed.

7. Concluding remarks

Within the scope of special functions in mathematics and physics, although the Rogers–Szegö polynomials play an important role in specific problems related to q-deformed algebras, they still remain practically unexplored if one considers the wide range of applications in quantum mechanics. A remarkable property of this sort of orthogonal polynomials states that $\{\mathscr{H}_n(z;q)\}_{n\in\mathbb{N}}$ are orthogonalized on the unit circle (pictorially represented in the complex plane) by means of a particular measure, the Jacobi ϑ_3-function [42]. In the quantum-mechanical context, the benefits of this kind of *angular representation*, if employed in the fascinating problem of the polar decomposition of the annihilation operator concerning the usual harmonic oscillator (or, in other words, on the existence of a well-defined operator corresponding to the phase observable of the electromagnetic field in quantum optics), were not investigated until now or even mentioned in the literature. Here, we have presented an appreciable set of new interesting results which have potential applications not only in the investigative process on the different polar decompositions of the q-deformed annihilation operator [48], where the aforementioned angular representation has an important role, but also in the study of a q-analogue to the Jordan–Schwinger mapping for the angular momentum operators [29].

Once the Szegö measure can be decomposed in the complex plane, it is natural to construct a set of complex functions that not only embodies such a decomposition but also satisfies automatically an orthogonality relation analogous to that derived for the RS polynomials. Before we perform such a task, some essential mathematical properties inherent to these polynomials have been adequately reviewed in our explanatory notes; in addition, we have also obtained two new integral representations for the RS polynomials which establish an important link with the q-Pochhammer symbol and the Stieltjes–Wigert polynomials [43]. In fact, such results have paved the way for subsequent developments towards a solid framework of coherent and phase states conceived within a purely algebraic approach.

Named as Rogers–Szegö functions and here denoted by $\left\{\Psi_n^{\mathrm{RS}}(z;q)\right\}_{n\in\mathbb{N}}$, our object of study was then defined in terms of the product $\mathscr{R}_n(z;q)\mathscr{M}(z;q)$, where the first term $\mathscr{R}_n(z;q)$ presents a direct connection with the RS polynomials, while the second term $\mathscr{M}(z;q)$ represents a complex weight function. The particular parametrization $z = -q^{-\frac{1}{2}}\mathrm{e}^{\mathrm{i}\varphi}$ with $\varphi \in [-\pi,\pi]$ has allowed us to verify that $\Psi_n^{\mathrm{RS}}(z;q)$ can be orthogonalized on a unit circle for any $q \in (0,1)$. Furthermore, we have also established a set of interesting formal results that characterize its inherent algebraic properties. For example, the discussion about the completeness relation pertaining to the RS functions was carried out by means of the bilinear kernel $K_{\varepsilon}(w,z;q)$, which obeys certain properties that lead us to reveal the normalization condition for the complex representations used in this work to describe such functions. In the following, we have analysed separately each term of $\Psi_n^{\mathrm{RS}}(z;q)$ with the aim of obtaining not only new recurrence relations for $\mathscr{R}_n(z;q)$ but also some scaling relations for $\mathscr{M}(z;q)$ that involve odd and even powers of the parameter q. As a by-product of this analysis, we have introduced a specific definition of Jackson's q-derivative that permits us to determine two closed-form expressions which connect the action of $\mathcal{D}_{q^2}$ over $\Psi_n^{\mathrm{RS}}(z;q)$ with the different excitation degrees n and $n \mp 1$, preserving, in its turn, the phase of the RS functions. So, the construction of q-differential forms of the lowering, raising and number operators from these results can be considered, at this stage, as an immediate process. Those differential representations were then interpreted as the particular realization of the IACK algebra which was characterized, within this context, through well-established commutation relations.

The applications of this algebraic approach in the construction process of coherent and phase states certainly represented an ideal scenario within our investigative theoretical framework. In this way, we adopted in a first stage the mathematical procedure developed by Barut–Girardello [51] for the coherent states with the main aim of establishing the respective eigenfunctions $\{\mathscr{F}_\mu(z;q)\}_{\mu\in\mathbb{C}}$ related to the lowering operator $\mathbf{B}$. Such eigenfunctions were initially conceived as an infinite expansion of the Rogers–Szegö functions whose coefficients satisfy a proper eigenvalue equation which leads us to obtain the excitation probability distribution for the q-deformed coherent state. Besides, we evaluated the overlap probability and also discussed the completeness relation in this case. Hounkonnou and Ngompe Nkouankam [30] recently carried out an interesting study on the generalized hypergeometric coherent states, where a (q,ν)-deformation was introduced in such a case. In that work, the authors employed basically some theoretical methods of quantum optics for investigating the quantum statistical properties as well as the Husimi distribution (and its corresponding phase distribution) of those particular coherent states. Such an analysis can also be applied within the context here discussed, this fact being the object of future investigations.

Finally, let us now briefly comment on the results obtained for the phase states. Basically, we have followed the Carruthers–Nieto approach [47] for the construction of the eigenstates related to the q-deformed cosine and sine operators, whose respective phase distributions have exhibited analogous behaviours but with distinct signatures (both the distributions are $\frac{\pi}{2}$-dephased) in the $\gamma\varphi$-plane. Next, we have discussed two basic properties inherent to the eigenstates $\{\mathscr{X}_\gamma(z;q)\}_{\gamma\in[0,\pi]}$ and $\{\mathscr{Y}_\gamma(z;q)\}_{\gamma\in[-\frac{\pi}{2},\frac{\pi}{2}]}$ which reflect their orthogonality and completeness (or resolution of unity) relations. In addition, we have applied our results in order to derive a set of closed-form expressions for certain mean values associated with the aforementioned cosine and sine operators via q-deformed coherent states, which allow us to study the Robertson–Schrödinger uncertainty relation. To complete this initial study, we have also derived a symmetrical uncertainty relation (here involving the variances and covariances of the cosine, sine and nondeformed number operators) that corroborates our previous conclusions on those coherent states: once the convergence criterion $0 \leqslant (1-q)|\mu|^2 < 1$ is satisfied, they are not minimum uncertainty states (except for the vacuum state $|\mu| = 0$).

Although the mathematical significance of q deformation is presently approached as being a parameter responsible for the distribution width, its physical appeal is not quite clear. In fact, even the squeezing and/or nonlinear effects attributed to q deserve to be investigated in detail [16]. Furthermore, the difficulties in solving the intriguing problem related to the phase operator in quantum mechanics still remain the same [49]. In this sense, it is worth stressing that the compilation of results here presented not only corroborates and generalizes those obtained in [25, 47], but also represents a concatenated effort in joining two promising research branches of mathematics and physics, namely the branches devoted to the study of q-special functions and certain angular representations in quantum theory [58].

In a more pragmatic sense, the previously discussed Rogers–Szegö functions can be seen to be tailored for describing deformed physical systems where the rotational degree of freedom has a central role. Indeed, some previous attempts at introducing rotational coherent states have been put forth in the past whose aim was to treat the dynamics of two-dimensional deformed systems in molecular physics [59]. In this connection, the use of the algebraic framework here developed in such studies of rigid deformed systems dynamics—within the context of von Neumann–Liouville formalism—seems to be a promising perspective. Moreover, our results also seem to be quite suitable to deal with the problem of quantum rings [60], where a single electron can be trapped in a region whose topology is exactly that covered in this work. In the meantime, it is worth stressing that there exists another path for future research which will be properly explored in due course.

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